Continuously varying exponents and the value of the central charge

1987 J. Phys. A: Math. Gen. 20 L891

(http://iopscience.iop.org/0305-4470/20/13/014)

J. Phys. A: Math. Gen. 20 (1987) L891-L896. Printed in the UK

LETTER TO THE EDITOR

Continuously varying exponents and the value of the central charge

John L Cardy†

Service de Physique Théorique, CEN Saclay, 91191 Gif-sur-Yvette, France

Received 2 July 1987

Abstract. We show that the conformal field theories describing two-dimensional critical behaviour with continuously varying exponents must have central charge $c = 1$ if (a) there are no conserved spin-2 currents other than the stress tensor and (b) the marginal operator responsible for the line of fixed points does not mix with other operators. In such cases one may show the existence of a fixed line knowing only the four-point function of the marginal operator. We apply this to the Ashkin-Teller model and the XY model with fourfold symmetry breaking.

Ever since Baxter's (1971) solution of his eponymous vertex model, it has been known that certain two-dimensional statistical mechanical models exhibit a line of critical points along which at least some of the critical exponents vary in a continuous fashion. Besides Baxter's model, these include the XY model (Kosterlitz and Thouless 1973), the Ashkin-Teller model (dual to the Baxter model), certain Ising models with three-spin interactions (Alcaraz and Barber 1987), magnetic hard squares (Pearce and Kim 1987) and spin-$\frac{1}{2}$ quantum chains. For all these examples it has been found either analytically, or with great numerical accuracy, that they correspond to conformal field theories with central charge $c = 1$. This is consistent with renormalisation group (RG) arguments which indicate, with varying degrees of rigour, that all these models renormalise onto the Gaussian model. It is also consistent with the result of Friedan *et al* (1984) that the critical exponents of reflection positive theories with $c < 1$ are quantised and therefore cannot vary continuously. This result leaves open the possibility of such theories existing with $c > 1$. So far, all exactly soluble examples of unitary conformal field theories with $c > 1$ either have discrete values for the exponents (Friedan *et al* 1984, Knizhnik and Zamolodchikov 1984, Fateev and Zamolodchikov 1985, 1986, 1987) or they decompose into a direct product of theories, at least some of which have $c = 1$. Thus, the present investigation was aimed at showing that all non-decomposable theories with continuously varying exponents must have $c = 1$. However, we have not succeeded in doing this, and in fact there is numerical evidence against this conjecture (von Gehlen and Rittenberg 1987). We can, however, show that continuously varying exponents arise more naturally in theories with $c = 1$, in the sense, to be explained more fully below, that when $c > 1$ they can only arise through a complicated conspiracy of the operator product expansion (OPE) coefficients.

† Permanent address: Department of Physics, University of California, Santa Barbara, CA 93106, USA.

We adopt the standard RG interpretation of continuously varying exponents (which we believe is the only one consistent with the assumed analyticity of RG flows) that they arise from the existence of a manifold of fixed points. For simplicity we take this to be one dimensinal, labelled by a parameter g. The fixed-point Hamiltonians are denoted by $\mathscr{H}(g)$. There is then a local marginal operator $\Phi(g)=\delta\mathscr{H}/\delta g$ with scaling dimensions $(1,1)$ all the way along the fixed line. This means that the two-point function $\langle\Phi(g;z_1,\bar{z}_1)\Phi(g;z_2,\bar{z}_2)\rangle$ behaves as $z_{12}^{-2}\bar{z}_{12}^{-2}$ for all g. Let us imagine that we calculate this in a perturbation expansion about $g=0$. Denoting $\Phi(g=0)$ by Φ, the $O(g)$ correction is $g\int d^2z\langle\Phi(g;z_1,\bar{z}_1)\Phi(g;z_2,\bar{z}_2)\Phi(z,\bar{z})\rangle$. This integral must be regulated with a cutoff $|z-z_1|>a$, $|z-z_2|>a$, but the result must be independent of a in order to give a conformally invariant result. The a dependence can be investigated using the OPE

$$\Phi(z,\bar{z})\Phi(0,0)\sim z^{-2}\bar{z}^{-2}\Big(1+bz\bar{z}\Phi(0,0)+\sum_i b_i(z\bar{z})^{x_i/2}\phi_i(0,0)\Big) \tag{1}$$

where, because of the integrations involved, we need include only the contribution of the scalar, non-derivative (quasiprimary, see Belavin *et al* (1984)) operators on the right-hand side. In (1) we have introduced the OPE coefficients b and b_i. Because of the orthogonality of quasiprimary operators (Belavin *et al* 1984) $\langle\Phi\phi_i\rangle=0$, and only the term in (1) proportional to Φ contributes to the three-point function. This will necessarily lead to logarithmic dependence on a (which has the effect of renormalising the scaling dimensions of Φ) unless $b=0$.

To $O(g^2)$ there is a contribution

$$\tfrac{1}{2}g^2\int d^2z\, d^2z'\,\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z,\bar{z})\Phi(z',\bar{z}')\rangle_{\text{conn}} \tag{2}$$

which should, once again, be cut off. Now, however, the other terms in (1) give contributions like

$$\frac{\pi b_i a^{x_i-2}}{y_i}\int d^2z\,\langle\Phi(z_1,\bar{z}_1)\phi_i(z_2,\bar{z}_2)\Phi(z,\bar{z})\rangle \tag{3}$$

where $y_i=2-x_i$, from, e.g., the region $|z'-z_2|=O(a)$. Note that the integral in (3) is itself proportional to b_i. Such a dependence can only be cancelled if we recognise that, to $O(g^2)$, the fixed-line Hamiltonian is not $\mathscr{H}(0)+g\Phi$, but must also contain a piece $\pi g^2\sum_i b_i\phi_i/y_i$. Insertions of this operator then cancel the a dependence of (3).

These results can also be seen by deriving the RG equations, which, to this order, are determined by the OPE coefficients (see e.g. Zamolodchikov 1986)

$$\begin{aligned} dg/dl &= -\pi b g^2+O(g^3)\\ dg_i/dl &= (2-x_i)g_i-\pi b_i g^2+O(g_i^2,g^3)\end{aligned} \tag{4}$$

from which we see that a fixed line is only possible if $b=0$, and that the equation of this line is $g_i=[\pi b_i/(2-x_i)]g^2+O(g^3)$. This means that, to $O(g)$, the marginal operator is not Φ but rather $\Phi+2\pi g\sum_i b_i\phi_i/y_i$. This additive renormalisation of Φ is sufficient to cancel the short-distance singularities of (2) when either z or z' approach z_1 or z_2 individually. However, there remain potential divergences when, for example, $|z-z_1|$ and $|z'-z_1|$ are both $O(a)$. To investigate this region, recall that conformal invariance implies that the four-point function has the form (Polyakov 1970)

$$|z-z_1|^{-4}|z'-z_2|^{-4}F_c\left(\zeta=\frac{(z'-z_1)(z-z_2)}{(z-z_1)(z'-z_2)},\bar{\zeta}\right) \tag{5}$$

which, in the region of interest, reduces to

$$|z-z_1|^{-4}|z_{12}|^{-4}F_c\left(\zeta=\frac{z'-z_1}{z-z_1},\bar{\zeta}\right).$$

Changing variables to z and ζ, the integral has the form

$$|z_{12}|^{-4}\int\frac{d^2z}{|z-z_1|^2}\int d^2\zeta\, F_c(\zeta,\bar{\zeta}). \tag{6}$$

Thus the only way to avoid a logarithmic divergence is to have $\int F_c(\zeta,\bar{\zeta})\,d^2\zeta=0$. This is a necessary condition for a fixed line to exist, and is equivalent to a condition on the $O(g^3)$ terms in the RG equations (4). The connected part F_c is related to the scaling function of the full four-point function F by

$$F_c(\zeta,\bar{\zeta})=F(\zeta,\bar{\zeta})-\frac{1}{|\zeta|^4}-\frac{1}{|1-\zeta|^4}-1 \tag{7}$$

where F obeys the crossing symmetry conditions (Belavin *et al* 1984)

$$F(\zeta,\bar{\zeta})=F(1-\zeta,1-\bar{\zeta})=|\zeta|^{-4}F(\zeta^{-1},\bar{\zeta}^{-1}). \tag{8}$$

According to Belavin *et al* (1984) F may also be written as a sum of squares of analytic functions:

$$F(\zeta,\bar{\zeta})=|F_{\rm I}(\zeta)|^2+\sum_i{}' b_i^2|F_i(\zeta)|^2 \tag{9}$$

representing the contributions of the different conformal blocks. The first term is the contribution of the conformal block of the identity, while the remaining sum over i is over primary operators only (to each primary operator there corresponds an infinite number of quasiprimary operators; however, the OPE coefficients b_i are determined for these once those of the primary operators are known). The functions $F_{\rm I}$ and F_i are completely determined by the value of c and the scaling dimensions.

Let us now assume that all the b_i appearing in (9) vanish. From our previous result, this implies the natural behaviour that the marginal operator $\Phi(g)$ mixes only with (irrelevant) operators in the conformal block of the identity as we move along the fixed line. In this case F is the square of a real analytic function $F_{\rm I}(\zeta)$ such that its only singularity in $|\zeta|<1$ is of the form

$$F_{\rm I}(\zeta)=\frac{1}{\zeta^2}+\text{regular terms.}$$

The coefficient of the most singular term is dictated by the form of the disconnected pieces in (7). There is no ζ^{-1} term since $L_{-1}1=0$. It is straightforward to show that the only such function whose modulus squared satisfies the crossing requirements (8) is

$$F_{\rm I}(\zeta)=F_{\rm G}(\zeta)\equiv\frac{1}{\zeta^2}+\frac{1}{(1-\zeta)^2}+1 \tag{10}$$

which implies that

$$F_c(\zeta,\bar{\zeta})=2\,\mathrm{Re}\left(\frac{1}{\zeta^2}+\frac{1}{(1-\zeta)^2}+\frac{1}{\bar{\zeta}^2(1-\zeta)^2}\right). \tag{11}$$

The condition $\int F_c\,d^2\zeta=0$ is then fulfilled provided the integral is defined via a principal value prescription which respects rotational invariance.

Knowing the exact form of the full four-point function $|F_G(\zeta)|^2$ we can now determine c (Belavin *et al* 1984). To see this, recall that the limit $\zeta \to 0$ also corresponds to $|z'-z_1| \sim |z-z_2| \ll |z-z_1| \sim |z'-z_2|$. In this region we can use the OPE equation (1):

$$\Phi(z_1, \bar{z}_1)\Phi(z, \bar{z}) \sim |z-z_1|^{-2}|\bar{z}-\bar{z}_1|^{-2}(1 + B(z-z_1)^2 T(z_1) + B(\bar{z}-\bar{z}_1)^2 \bar{T}(\bar{z}_1) + \ldots) \qquad (12)$$

where we have now included the leading non-scalar operators, the components T, $\bar{T}$ of the stress tensor. The coefficient B is determined by forming the correlation function of (12) with $T(z_2)$, which is fixed by the conformal Ward identity, and by using $\langle T(z_1)T(z_2)\rangle = c/2(z_1-z_2)^4$. This gives $B = 2/c$. The $O(\zeta^{-2})$ term in $F(\zeta, \bar{\zeta})$ then comes from the term proportional to $(z-z_1)^2(z'-z_2)^2$ when we insert (12) into the four-point function. Thus $F(\zeta, \bar{\zeta}) \sim |\zeta|^{-4} + (c/2)B^2(\zeta^{-2} + \bar{\zeta}^{-2}) + \ldots$. Comparison with (10) shows that $c = 1$.

This argument fails if there exist other spin-2 conserved currents besides $(T, \bar{T})$, since they will in general appear in (12) and hence contribute to the $O(\zeta^{-2})$ term in $F(\zeta, \bar{\zeta})$. In this case, one may show from the above argument only that $c \geq 1$, in a unitary theory. Zamolodchikov (1985) has shown that theories with N additional conserved spin-2 currents have the symmetry of $2N+2$ commuting Virasoro algebras. If the marginal operator has weights $(h_i, \bar{h}_i)$ with respect to the ith pair, then the condition on $F(\zeta, \bar{\zeta})$ implies that

$$\frac{2}{c} + 2\sum_{i=1}^{N} \frac{h_i^2}{c_i} = 2 \qquad (13)$$

where c_i is the central charge of the ith pair. Thus only if the marginal operator has weight zero with respect to the other Virasoro algebras can we conclude that $c = 1$.

The form (10) can be shown by a simple calculation to be valid in the Gaussian model, with Hamiltonian $\mathscr{H} = (g/4\pi)\int (\partial\phi)^2\, d^2z$, for which $\Phi \propto (\partial\phi)^2$. For this model we know that there exists a fixed line, i.e. Φ is marginal to all orders in g. This may be used to show that *any* theory for which $b = 0$ and for which the four-point function of the marginal operator is given by $|F_G(\zeta)|^2$ has a fixed line.

For such a theory all the OPE coefficients b_i (ϕ_i not in the conformal block of the identity) vanish. We may now imagine calculating the n-point function $\langle \Phi(0)\Phi(z_1)\Phi(z_2)\ldots\rangle$ in the region $|z_1| < |z_2| < \ldots$ using the OPE. Only 'intermediate states' corresponding to operators in the conformal blocks [1] and [Φ] will appear. Thus the n-point function will be identical to the n-point function of $(\partial\phi)^2$ in the Gaussian model. Since the marginality of Φ to $O(g^{n-2})$ involves the vanishing of logarithmic divergences in an integral over the n-point function, and since we know that these divergences do not appear in the Gaussian model, this proves our statement.

We can apply this result to two well known examples. The Ashkin–Teller model consists of two Ising models coupled via their energy densities ε_1 and ε_2. Thus, at the decoupling point $\Phi = \varepsilon_1\varepsilon_2$. In this case, the vanishing of b follows from the vanishing of $\langle \varepsilon_1\varepsilon_1\varepsilon_1\rangle$ by duality. The scaling form of the four-point function is simply the square of that for $\langle \varepsilon_1\varepsilon_1\varepsilon_1\varepsilon_1\rangle$:

$$F(\zeta, \bar{\zeta}) = \left|1 - \frac{1}{\zeta} - \frac{1}{1-\zeta}\right|^4 = \left|1 + \frac{1}{\zeta^2} + \frac{1}{(1-\zeta)^2}\right|^2. \qquad (14)$$

It follows without further calculation that the Ashkin–Teller model has a fixed line. As a second example, take the Gaussian model defined above, with the field ϕ living on a circle of radius one. In that case the primary operators $O_{n,m}$, with integer electric and magnetic charges n and m respectively, have scaling dimensions $x_{n,m} = n^2/2g + gm^2/2$, and spin nm. For a denumerable number of values of g there exists an additional marginal operator with $x = 2$. However, we may calculate its four-point function and find that it is not the square of an analytic function, so it does not generate a fixed line. But there are three values of g $(4, 1, \frac{1}{4})$ where there are (at least) two additional marginal operators. For example at $g = 4$ (the Kosterlitz-Thouless point) $O_{\pm 4,0}$ and $O_{0,\pm 1}$ are marginal. If we form the operator $\Phi = O_{4,0} + 0_{-4,0} + O_{0,1} + O_{0,-1}$ we find that the interference terms in its four-point function just conspire to give $|F_G(\zeta)|^2$. Thus we know this operator generates a fixed line. This fixed line occurs at the self-dual point of an XY model with fourfold symmetry breaking. Its existence was first conjectured by José *et al* (1977) and Kadanoff (1977) and checked to second order by Zisook (1980).

There are other fixed lines emerging from this point, and the $g = 1$ point is even richer, having SU(2)⊗SU(2) symmetry. We intend to pursue this in a further work.

In all the above, we made the simplifying assumption that $b_i = 0$ $(\phi_i \not\in [1])$. If this is not true, then the condition on the four-point function can be written

$$\int \left(|F_1(\zeta)|^2 - |F_G(\zeta)|^2 + \sum_i b_i^2 |F_i(\zeta)|^2 \right) d^2\zeta = 0. \tag{15}$$

The first two terms cancel when $c = 1$, leading to $b_i = 0$. Alternatively we see that if $c \neq 1$ then some $b_i \neq 0$. If this is the case then the conditions on the higher-order correlation functions become non-trivial. In particular they involve all the OPE coefficients, not just the b_i. Thus a fixed line can only appear in such theories by some conspiracy of the OPE coefficients (which may, of course, appear due to an unsuspected symmetry).

In calculations in finite-width strips, the marginal operator should correspond to a state with an energy gap $4\pi/L$ (Cardy 1984). The corrections to this behaviour due to irrelevant operators will be (Cardy 1986) proportional to $b_i L^{1-x_i}$. In theories with $c > 1$ we therefore expect the corrections to the energy gap to be larger. For theories with $c = 1$ we have shown that $b_i = 0$ $(\phi_i \not\in [1])$ which means that the leading corrections come from operators with $x = 4$.

I thank J B Zuber, C Itzykson, E Rabinovici, G von Gehlen and V Rittenberg for useful discussions.

References

Alcaraz F C and Barber M N 1987 *J. Stat. Phys.* to be published
Baxter R J 1971 *Phys. Rev. Lett.* **26** 832
Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333
Cardy J L 1984 *J. Phys. A: Math. Gen.* **17** L385
—— 1986 *Nucl. Phys.* B **270** [FS16] 186
Fateev V A and Zamolodchikov A B 1985 *Sov. Phys.-JETP* **62** 215
—— 1986 *Sov. Phys.-JETP* **63** 913
—— 1987 *Nucl. Phys.* B **280** 644
Friedan D, Qui Z and Shenker S 1984 *Phys. Rev. Lett.* **52** 1575
José J V, Kadanoff L P, Kirkpatrick S and Nelson D R 1977 *Phys. Rev.* B **16** 1217
Kadanoff L P 1977 *Phys. Rev. Lett.* **39** 903
Knizhnik V and Zamolodchikov A B 1984 *Nucl. Phys.* B **247** 83
Kosterlitz J M and Thouless D J 1973 *J. Phys. C: Solid State Phys.* **6** 1181

Pearce P A and Kim D 1987 *J. Phys. A: Math. Gen.* to be published
Polyakov A M 1970 *JETP Lett.* **12** 381
von Gehlen G and Rittenberg V 1987 private communication
Zamolodchikov A B 1985 *Theor. Math. Phys.* **65** 347
—— 1986 *JETP Lett.* **43** 565
Zisook A B 1980 *J. Phys. A: Math. Gen.* **13** 2451